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State-constrained optimal control of phase-field equations with obstacle

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Abstract

This paper is concerned with an optimal control problem for the phase-field transition system with state constraint and obstacle. After showing the relationship between the control problem and its approximation, we derive Pontryagin's maximum principle for an optimal control of our original problem by using one of the approximate problems.

MSC: 49K20; 49J20; 74N25**Keywords:** phase-field transition system; optimal control; Pontryagin's maximum principle

1 Introduction

We consider the optimal control of solid-liquid phase transitions:

$$\begin{cases} u_t - \gamma \Delta u + \delta \partial I_{[-1,1]}(u) + \varphi(u) - \lambda'(u)v \ni 0 & \text{in } Q = \Omega \times (0, T), \\ v_t + (\lambda(u))_t - k \Delta v = Bw & \text{in } Q = \Omega \times (0, T), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma = \partial \Omega \times (0, T), \end{cases} \quad (1.1)$$

and

$$F(u) \subset S,$$

where Ω is a bounded domain in R^N ($1 \leq N \leq 3$) with a smooth boundary $\partial \Omega$, $\varphi(u) = u^3 - u$, $\delta, \gamma, k > 0$ denote given parameters, $\partial I_{[-1,1]}(u)$ is the subdifferential of the indicator function $I_{[-1,1]}(u)$ on the closed interval $[-1, 1]$, Bw is a given forcing term on Q , $\frac{\partial}{\partial \nu}$ is the outward normal derivative on $\partial \Omega$ and u_0, v_0 are given initial datums.

System (1.1) is a simplified model for a class of solid-liquid phase change problems. In the context of solid-liquid phase transitions, v and u represent the absolute temperature and the order parameter which indicates the physical situation of the system, respectively. Therefore it is natural to assume that the range of u is bounded, say the closed interval $[-1, 1]$ in this paper, and $\partial I_{[-1,1]}(u)$ denoting the range of the order parameter u is assumed to be a compact interval $[-1, 1]$, $u(t, x) \equiv -1$ and $u(t, x) \equiv 1$ mean, respectively, that the physical situation at (t, x) is of pure solid and pure liquid, while $-1 < u(t, x) < 1$ means that the physical situation at (t, x) is mushy.

A great deal of research has been done on the phase-field transition system, for which we refer to the book by Temam [1] and the references therein. Without the term $\partial I_{[-1,1]}(u)$, system (1.1) is the standard phase field model which was studied in [2, 3]. One of the most important characteristics of our model is the nonlinear term $\partial I_{[-1,1]}(u)$ (obstacle) which allows the coexistence of pure phases in the dynamical phase transition process. The existence and uniqueness of solution for the phase field model with obstacle were discussed in [4–8]. In particular, the asymptotic behavior of solutions to the non-isothermal phase-field transition system with obstacle was considered in [6] and [7]. Recently, the Caginalp phase-field system with coupled dynamic boundary conditions, including the singular potentials, was presented in [9] and [10].

Throughout this paper, the Hilbert space $H = L^2(\Omega)$ is equipped with the usual inner product (\cdot, \cdot) and the norm $\|\cdot\|_2$. Define a closed subspace H_0 of H by $H_0 = \{z \in H; \int_{\Omega} z \, dx = 0\}$. We put $V_0 = V \cap H_0$ with $\|v\|_{V_0} = \|\nabla v\|_{L^2(\Omega)} \equiv \|v\|_2$, where $V = \{v \in H^1(\Omega), \frac{\partial v}{\partial \nu} = 0\}$. If we identify H^* and H_0^* with their dual space, then we have $V \subset H \subset V^*$ and $V_0 \subset H_0 \subset V_0^*$, where V_0^* is the duality space of V_0 .

Throughout the paper, we suppose that the following assumptions hold.

Let U be a real Hilbert space, $B: U \rightarrow H$ be a linear continuous operator. Assuming that Z is a Banach space with the dual Z^* strictly convex, let $S \subset Z$ be a closed convex subset with finite co-dimensionality.

(H₁) $F: L^2(0, T; H) \rightarrow Z$ is in the class of C^1 .

(H₂) $g: [0, T] \times H \rightarrow \mathbb{R}^+$ is measurable in t and for every $\sigma > 0$, there exists $L_{\sigma} > 0$ independent of t such that $g(0, u) \in L^{\infty}(0, T)$ and

$$|g(t, y) - g(t, z)| \leq L_{\sigma} |y - z|_2 \quad \text{for any } t \in [0, T] \text{ with } |y|_2 + |z|_2 \leq \lambda.$$

(H₃) $h: U \rightarrow \bar{R}$ is lower semicontinuous and convex with the following growth property:

$$h(w) \geq c_1 |w|_U^2 + c_2 \quad \text{for any } w \in U \text{ with } c_1 > 0 \text{ and } c_2 \in R.$$

(H₄) $\lambda \in C^2$ and there exists a constant $\kappa > 0$ such that $\lambda''(s) \leq \kappa$ for any $s \in R$.

We consider the following optimal control problem:

$$\min L(w) \quad \text{over all } w \in L^2(0, T; U), \tag{P}$$

where

$$L(w) = \int_0^t [g(t, u(t)) + h(w(t))] \, dt$$

and

(u, v) is the solution of (1.1) corresponding to w , $F(u) \subset S$.

For any

$$(u, v, w) \in Y \times Y \times L^2(0, T; U)$$

satisfying (1.1) is called a feasible pair, where $Y = H^{2,1}(Q) \cap C(0, T; V)$.

The first question regarding problem (P) is if there is an admissible solution, *i.e.*, if the set A_{ad} is nonempty. Taking into account [11] similarly, we may assume in the sequel that for $(u, v, w) \in Y \times Y \times L^2(0, T; U)$, problem (P) admits at least one admissible solution.

Optimal control problems of the phase transition system have been studied by several authors (for instance, see [12–16]). In particular, let $\lambda(u) = \frac{l}{2}$ and $\delta = 0$ in (1.1), the optimal boundary controls for a phase field model and the state-constrained optimal control for the phase-field transition system were considered in [13] and [15], respectively. In [16], based on the energy estimates and the compact method, Ryu and Yagi considered the optimal control problems of the adsorbate-induced phase transition model. It is noted that the optimal control without state constraint or without obstacle of the phase field model was discussed in [14, 17–20].

To the best of our knowledge, there are few papers concerned with the optimal control problems for the phase-field with obstacle although it is natural to have the obstacle in the solid-liquid phase transitions and related physics models, since the obstacle $\partial I_{[-1,1]}(u)$ brings the essential difficulty in getting Pontryagin's maximum principle for corresponding models.

We state the maximum principle as follows.

Theorem 1.1 *Suppose that (H_1) , (H_2) , (H_3) and (H_4) hold. Let (u^*, v^*, w^*) be optimal for problem (P), then there exists a tetrad $(\mu_0, p, q, \zeta_0) \in R \times L^2(0, T; V) \cap L^\infty(0, T; H) \times L^2(0, T; V) \cap L^\infty(0, T; H) \times Z^*$ with $(\mu_0, \zeta_0) \neq 0$ and a measure $\eta \in L^\infty(Q)^*$ such that*

$$\begin{cases} -p_t - \gamma \Delta p + \eta + (3(u^*)^2 - 1)p - \lambda''(u^*)pv^* - \lambda'(u^*)q \\ \quad \in -[\partial F(u^*)]^* \zeta_0 - \mu_0 \partial g(t, u^*), \\ -q_t + \lambda'(u^*)p_t - k \Delta q + \lambda''(u^*)pu_t^* = 0, \\ p(T) = 0, \quad q(T) = 0 \end{cases} \quad (1.2)$$

and

$$\begin{cases} B^* q(t) \in \mu_0 \partial h(w^*(t)), \\ \langle \zeta_0, s - \partial F(u^*) \rangle_{Z^*, Z} \leq 0 \quad \forall s \in S. \end{cases} \quad (1.3)$$

Moreover, if $F'(u^*)$ is injective, then $(\mu_0, p, q) \neq 0$.

The rest of this paper is organized as follows. In Section 2, we provide existence results and *a priori* estimates in the form that is required to obtain Pontryagin's maximum principle for problem (P). Besides the existence of an optimal control in problem (P^ε) , necessary optimality conditions for this problem and for problem (P) are proved in Section 3.

2 The approximation problem

This section is to show the existence of the optimal control of the approximation problem corresponding to the phase transition system. To this end, we first show some technical lemmas, which are presented below for the sake of completeness.

In order to approximate $\partial I_{[-1,1]}(\cdot)$, we define a nondecreasing function β^ε [21] on R by

$$\beta^\varepsilon(r) = \text{sign}(r) \int_0^{|r|} \min \left\{ \frac{1}{\varepsilon}, \frac{[s-1]^+}{\varepsilon^2} \right\} ds \quad \forall r \in R,$$

where $[\cdot]^+$ denotes the positive part of functions. Then $\beta^\varepsilon \in C^1$, $(\beta^\varepsilon)' \in W^{1,\infty}(R)$ and

$$0 \leq (\beta^\varepsilon)'(r) \leq \frac{1}{\varepsilon}, \quad |(\beta^\varepsilon)'(r)| \geq \frac{1}{\varepsilon}([r-1]^+ + [-1-r]^+) - \frac{1}{2} \quad \text{for any } r \in R. \quad (2.1)$$

We fix a primitive $\hat{\beta}^\varepsilon$ of β^ε such that

$$\hat{\beta}^\varepsilon(0) = 0 \quad \text{and} \quad \hat{\beta}^\varepsilon(r) \geq 0 \quad \text{for any } r \in R. \quad (2.2)$$

Without loss of generality, we may assume $\delta = 1$, therefore, the approximation of (1.1) is

$$\begin{cases} u_t - \gamma \Delta u + \beta^\varepsilon(u) + \varphi(u) - \lambda'(u)v = 0 & \text{in } Q = \Omega \times (0, T), \\ v_t + \lambda'(u)u_t - k \Delta v = Bw & \text{in } Q = \Omega \times (0, T), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma = \partial \Omega \times (0, T). \end{cases} \quad (2.3)$$

Lemma 2.1 Suppose that $\beta^\varepsilon(\cdot)$ satisfies (2.1)-(2.2), $w_n \in L^2(0, T; U)$, $w_n \rightarrow \tilde{w}$ weakly in $L^2(0, T; U)$ and (\tilde{u}, \tilde{v}) , (u_n, v_n) are the solutions of (2.3) corresponding to \tilde{w} and w_n , respectively. Then there exists a subsequence of (u_n, v_n) , still denoted by itself, such that

$$\begin{aligned} (u_n, v_n) &\rightarrow (\tilde{u}, \tilde{v}) \quad \text{weakly in } Y^2, \\ \beta^\varepsilon(u_n) &\rightarrow \tilde{\eta} \quad \text{weakly in } L^2(0, T; H) \text{ as } n \rightarrow \infty. \end{aligned} \quad (2.4)$$

Proof Replacing (u, v) and w by (u_n, v_n) and w_n in (2.3), respectively, we obtain

$$\begin{cases} u_{n,t} - \gamma \Delta u_n + \beta^\varepsilon(u_n) + \varphi(u_n) - \lambda'(u_n)v_n = 0 & \text{in } Q = \Omega \times (0, T), \\ v_{n,t} + \lambda'(u_n)u_{n,t} - k \Delta v_n = Bw_n & \text{in } Q = \Omega \times (0, T), \\ u_n(x, 0) = u_0(x), \quad v_n(x, 0) = v_0(x) & \text{in } \Omega, \\ \frac{\partial u_n}{\partial \nu} = \frac{\partial v_n}{\partial \nu} = 0 & \text{on } \Sigma = \partial \Omega \times (0, T). \end{cases} \quad (2.5)$$

Multiplying (2.5)₁ and (2.5)₂ by $u_{n,t}$ and v_n , respectively, integrating over Ω and adding the resulting equations, we end up with

$$\begin{aligned} &\frac{d}{dt} \left(\frac{\gamma}{2} |\nabla u_n|_2^2 + |v_n|_2^2 \right) + |u_{n,t}|_2^2 + k |\nabla v_n|_2^2 + \int_{\Omega} (\beta^\varepsilon(u_n) + \varphi(u_n)) u_{n,t} dx \\ &\leq - \int_{\Omega} Bw_n v_n dx. \end{aligned} \quad (2.6)$$

Therefore, we conclude with the help of Young's inequality and the properties of β^ε that

$$\begin{aligned} &\frac{d}{dt} \left(\frac{\gamma}{2} |\nabla u_n|_2^2 + |v_n|_2^2 + \frac{1}{4} |u_n|_{L^4}^4 + |\hat{\beta}^\varepsilon|_{L^1} \right) + k |\nabla v_n|_2^2 + |u_{n,t}|_2^2 \\ &\leq \frac{1}{4} |Bw_n|_2^2 + |v_n|_2^2 + \frac{1}{2} |u_{n,t}|_2^2 + \frac{1}{2} |u_n|_2^2 \\ &\leq \frac{1}{4} |Bw_n|_2^2 + |v_n|_2^2 + \frac{1}{2} |u_{n,t}|_2^2 + |u_n|_{L^4}^4 + C. \end{aligned} \quad (2.7)$$

Here and throughout the proof of Lemma 2.1, we shall denote by C several positive constants independent of n . Applying Gronwall's inequality to (2.7), we derive

$$|u_n|_{L^\infty(0,T;V \cap L^4)} + |v_n|_{L^\infty(0,T;H)} + |v_n|_{L^2(0,T;V)} + |u_n|_{H^1(0,T;H)} + |\hat{\beta}^\varepsilon|_{L^\infty(0,T;L^1)} \leq C. \quad (2.8)$$

Now, testing (2.5)₁ by $-\Delta u_n$, we derive

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |\nabla u_n|_2^2 + \gamma |\Delta u_n|_2^2 + \int_{\Omega} (\beta^\varepsilon)' |\nabla u_n|^2 + 3u_n^2 |\nabla u_n|^2 dx \\ &= - \int_{\Omega} \lambda'(u_n) v_n \Delta u_n dx \\ &\leq \frac{\gamma}{2} |\Delta u_n|_2^2 + C \left(|v_n|_{L^6}^6 + \int_{\Omega} |\lambda'(u_n)|^3 dx \right), \end{aligned} \quad (2.9)$$

which together with $\lambda''(t) \leq \kappa$, (2.2), (2.8), Nirenberg's inequality and Gronwall's inequality implies that

$$|u_n|_{L^\infty(0,T;V)} + |u_n|_{L^2(0,T;H^2)} \leq C. \quad (2.10)$$

Next, multiplying (2.5)₁ by β^ε , integrating over $[0, T]$ and invoking Young's inequality, we derive

$$\begin{aligned} & \frac{d}{dt} |\hat{\beta}^\varepsilon|_{L^1} + |\beta^\varepsilon|_2^2 \\ &= \int_{\Omega} (\gamma \Delta u_n - \varphi(u_n) + \lambda'(u_n) v_n) \beta^\varepsilon(u_n) dx \\ &\leq \frac{1}{2} |\beta^\varepsilon|_2^2 + C \left(|u_n|_{L^6}^6 + |u_n|_2^2 + \gamma^2 |\Delta u_n|_2^2 + \int_{\Omega} |\lambda'(u_n)|^3 dx + |v_n|_{L^6}^6 \right). \end{aligned} \quad (2.11)$$

Thanks to (2.8), $\lambda''(t) \leq \kappa$ and Gronwall's inequality, we derive

$$|\hat{\beta}^\varepsilon|_{L^\infty(0,T;L^1)} + |\beta^\varepsilon|_{L^2(0,T;H)} \leq C. \quad (2.12)$$

Inserting (2.10) and (2.12) into (2.5)₁, we have

$$|u_{n,t}|_{L^2(0,T;H)} \leq C. \quad (2.13)$$

Now, differentiating (2.5)₁ with respect to t and multiplying the result by $u_{n,t}$, then multiplying (2.5)₂ by v_t , adding the resulting equations and integrating over Ω leads to

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} |u_{n,t}(t)|_2^2 + \frac{k}{2} |\nabla v_n(t)|_2^2 \right) + \gamma |\nabla u_{n,t}|_2^2 + \int_{\Omega} (\beta^\varepsilon)' u_{n,t}^2(x, t) dx + |v_{n,t}|_2^2 \\ &\leq |u_{n,t}(t)|_2^2 + \int_{\Omega} \kappa v_n(t) u_{n,t}^2(t) dx + \int_{\Omega} B w_n v_{n,t} dx \\ &\leq |u_{n,t}(t)|_2^2 + \int_{\Omega} |\kappa v_n(t) u_{n,t}^2(t)| dx + \frac{1}{2} |v_{n,t}(t)|_2^2 + \frac{1}{2} |B w_n|_2^2. \end{aligned} \quad (2.14)$$

On the other hand, with the help of (2.8), (2.10), Hölder's inequality and Nirenberg's inequality, we get

$$\begin{aligned}
 & \left| \int_0^t \int_{\Omega} \kappa v_n(t) u_{n,t}^2 dx \right| \\
 & \leq |\kappa| \int_0^t |v_n(s)|_{L^6} |u_{n,t}(s)|_{L^{\frac{12}{5}}}^2 ds \\
 & \leq |\kappa| \left(\int_0^t |v_n(s)|_{L^6}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t |u_{n,t}(s)|_{L^{\frac{12}{5}}}^4 ds \right)^{\frac{1}{2}} \\
 & \leq C_1 \left(\int_0^t |v_n(s)|_{H^1}^2 ds \right)^{\frac{1}{2}} \times \left(\int_0^t (|\nabla u_{n,t}(s)| |u_{n,t}(s)|^3 + |u_{n,t}(s)|^4) ds \right)^{\frac{1}{2}} \\
 & \leq C_2 \sup_{0 \leq s \leq t} |u_{n,t}(s)| \times \left(\int_0^t (|\nabla u_{n,t}(s)| |u_{n,t}(s)| + |u_{n,t}(s)|^2) ds \right)^{\frac{1}{2}} \\
 & \leq C_3 \sup_{0 \leq s \leq t} |u_{n,t}(s)| \left[1 + \left(\int_0^t |\nabla u_{n,t}(s)|^2 ds \right)^{\frac{1}{4}} \right] \\
 & \leq \mu \sup_{0 \leq s \leq t} |u_{n,t}(s)|_2^2 + \mu \int_0^t |\nabla u_{n,t}(s)|^2 ds + C_4,
 \end{aligned} \tag{2.15}$$

where μ is a small positive constant and C_i ($i = 1, 2, 3, 4$) are independent of n . Inserting (2.15) into (2.14), we derive

$$\begin{aligned}
 & \frac{1}{2} |u_{n,t}(t)|_2^2 + \frac{k}{2} |\nabla v_n(t)|_2^2 \\
 & \quad + (\gamma - \mu) |\nabla u_{n,t}(t)|_2^2 + \int_0^t \int_{\Omega} (\beta^\varepsilon)'(u_n(t)) u_{n,t}^2 dx ds \\
 & \leq \int_0^t |u_{n,s}(s)|_2^2 ds + \mu \sup_{0 \leq s \leq t} |u_{n,t}(s)|_2^2 + \frac{1}{2} \int_0^t |Bw_n(s)|_2^2 ds + C.
 \end{aligned} \tag{2.16}$$

Taking the supremum with respect to t in (2.16), choosing $\mu > 0$ sufficiently small and applying Gronwall's inequality, we end up with

$$\sup_{0 \leq s \leq t} |u_{n,t}(t)|_2 + \sup_{0 \leq s \leq t} |\nabla v_n(t)|_2 + |\nabla u_{n,t}(t)|_2^2 \leq C_T, \tag{2.17}$$

which combined with (2.13) implies that $\lambda'(u_n)u_{n,t} \in L^2(0, T; H)$. Therefore, employing the standard parabolic theory to (2.5)₂ leads to

$$|v_{n,t}|_{L^2(0,T;H)} + |v_n|_{L^\infty(0,T;V)} + |v_n|_{L^2(0,T;H^2)} \leq C. \tag{2.18}$$

Now we may combine the estimates (2.8), (2.10), (2.12), (2.13) and (2.18) to conclude the results. This completes the proof. \square

Lemma 2.2 Suppose that $\beta^\varepsilon(\cdot)$ satisfies (2.1)-(2.2), let $w_\varepsilon \in L^2(0, T; U)$ with $w_\varepsilon \rightarrow w^*$ weakly in $L^2(0, T; U)$ as $\varepsilon \rightarrow 0$, $(u_\varepsilon, v_\varepsilon)$ be the solution of (2.3) corresponding to w_ε . Then, on some subsequence $(u_{\varepsilon_n}, v_{\varepsilon_n})$ of $(u_\varepsilon, v_\varepsilon)$, there exists a quad $(u, v, \eta) \in Y \times Y \times L^2(0, T; H)$

such that

$$\eta \in \partial I_{[-1,1]}(u) \quad \text{a.e. } L^2(0, T; H), \quad (2.19)$$

while

$$(u_{\varepsilon_n}, v_{\varepsilon_n}) \rightarrow (u, v) \quad \text{weakly in } (L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)))^2, \quad (2.20)$$

$$(u_{\varepsilon_n}, v_{\varepsilon_n}) \rightarrow (u, v) \quad \text{strongly in } (L^2(0, T; V) \cap C(0, T; H))^2, \quad (2.21)$$

$$(u'_{\varepsilon_n}, v'_{\varepsilon_n}) \rightarrow (u', v') \quad \text{weakly in } (L^2(0, T; H))^2, \quad (2.22)$$

$$\beta^\varepsilon(u_{\varepsilon_n}) \rightarrow \eta \quad \text{weakly in } L^2(0, T; H) \quad (2.23)$$

as $\varepsilon_n \rightarrow 0$ and (u, v, η) is a solution of (1.1) satisfying the following estimates:

$$|u|_Y^2 + |v|_Y^2 + |\eta|_{L^2(Q_T)}^2 \leq C, \quad (2.24)$$

where $C > 0$ is independent of ε, n .

Proof Rewrite (2.3) as follows:

$$\begin{cases} u_{\varepsilon,t} - \gamma \Delta u_\varepsilon + \beta^\varepsilon(u_\varepsilon) + \varphi(u_\varepsilon) - \lambda'(u_\varepsilon)v_\varepsilon = 0 & \text{in } Q = \Omega \times (0, T), \\ v_{\varepsilon,t} + \lambda'(u_\varepsilon)u_{\varepsilon,t} - k \Delta v_\varepsilon = Bw_\varepsilon & \text{in } Q = \Omega \times (0, T), \\ u_\varepsilon(x, 0) = u_0(x), \quad v_\varepsilon(x, 0) = v_0(x) & \text{in } \Omega, \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0 & \text{on } \Sigma = \partial\Omega \times (0, T). \end{cases} \quad (2.25)$$

Employing almost exactly the same arguments as in the proof of Lemma 2.1, we conclude the results (2.20)-(2.22). Furthermore, by a standard argument in [22], we get $\eta \in \partial I_{[-1,1]}(u)$ a.e. in $L^2(0, T; H)$. This completes the proof. \square

Now, we assume that (u^*, v^*, w^*) is optimal for problem (P). For each $\varepsilon > 0$, let $(u_\varepsilon^*, v_\varepsilon^*, w_\varepsilon^*)$ be the solution to

$$\begin{cases} u_t - \gamma \Delta u + \beta^\varepsilon(u) + \varphi(u) - \lambda'(u)v = 0 & \text{in } Q = \Omega \times (0, T), \\ v_t + \lambda'(u)u_t - k \Delta v = Bw^* & \text{in } Q = \Omega \times (0, T), \\ u(x, 0) = u_0^*(x), \quad v(x, 0) = v_0^*(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma = \partial\Omega \times (0, T). \end{cases} \quad (2.26)$$

It follows from Lemma 2.2 that

$$\delta(\varepsilon) \equiv \|u_\varepsilon^* - u^*\|_{L^2(0, T; H)} \rightarrow 0. \quad (2.27)$$

Now, the approximating optimal control problems (P^ε) are as follows:

$$\text{Minimize } L_\varepsilon(w) \quad \text{over } w \in L^2(0, T; U),$$

where $L_\varepsilon : L^2(0, T; U) \rightarrow R$, by

$$L_\varepsilon(w) = \int_0^T [g_\varepsilon(t, u_\varepsilon) + h(w)] dt + \frac{1}{2} \|w - w^*\|_{L^2(0, T; U)}^2 + \frac{1}{2\delta(\varepsilon)} [d_S(F(u_\varepsilon)) + \delta(\varepsilon)]^2 \quad (2.28)$$

and (u, v) is the solution of (2.3). Here, $d_S(F(u))$ denotes the distance of $F(u)$ to S ,

$$g_\varepsilon(t, y) = \int_{R^n} g(t, P_n y - \varepsilon \Lambda_n \tau) \rho_n(\tau) d\tau \quad (2.29)$$

is the approximations of g [23], where $n = [\frac{1}{\varepsilon}]$, ρ_n is a mollifier in R^n , $P_n : H \rightarrow X_n$ is the projection of H on X_n , which is the finite dimensional space generated by $\{e_i\}_{i=1}^n$, $\{e_i\}_{i=1}^\infty$ is an orthonormal basis in H , $\Lambda_n : R^n \rightarrow X_n$ is the operator defined by $\Lambda_n(\tau) = \sum_{i=1}^n \tau_i e_i$, $\tau = (\tau_1, \tau_2, \dots, \tau_n)$.

First of all, we show the existence of optimal solutions for (P^ε) .

Lemma 2.3 (P^ε) has at least one optimal solution.

Proof Let $\varepsilon > 0$ be fixed. It is clear that $\inf L_\varepsilon(w) > -\infty$. Let $d_\varepsilon = \inf\{L_\varepsilon(w) : w \in L^2(0, T; U)\}$ and w_n be a minimizing sequence such that

$$d_\varepsilon \leq L_\varepsilon(w_n) \leq d_\varepsilon + \frac{1}{n}, \quad (2.30)$$

which together with (H_2) , (H_3) and (2.28) implies that w_n is bounded in $L^2(0, T; U)$. Without loss of generality, we may assume that $w_n \rightarrow \tilde{w}$ in $L^2(0, T; U)$. Let (u_n, v_n) and (\tilde{u}, \tilde{v}) be the solutions of (2.3) corresponding to w_n and \tilde{w} , respectively. It follows from Lemma 2.1 that on some subsequence of (u_n, v_n) , still denoted by itself,

$$\begin{aligned} (u_n, v_n) &\rightarrow (\tilde{u}, \tilde{v}) \quad \text{weakly in } Y \times Y \quad \text{and} \\ &\text{strongly in } (C(0, T; H^1(\Omega)) \cap L^2(0, T; H))^2. \end{aligned} \quad (2.31)$$

With the help of (H_2) , (2.29) and (2.31), we also obtain

$$\int_0^T |g_\varepsilon(t, u_n) - g_\varepsilon(t, \tilde{u})|_{L^2} dt \leq C \int_0^T |u_n - \tilde{u}|_{L^2} dt \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.32)$$

On the other hand, due to (2.31) and (H_1) , we have that

$$\lim_{n \rightarrow \infty} F(u_n) = F(\tilde{u}) \quad (2.33)$$

and therefore

$$\lim_{n \rightarrow \infty} \frac{1}{2\delta(\varepsilon)} [d_S(F(u_n)) + \delta(\varepsilon)]^2 = \frac{1}{2\delta(\varepsilon)} [d_S(F(\tilde{u})) + \delta(\varepsilon)]^2. \quad (2.34)$$

Finally, (2.28) and (2.32)-(2.34) imply that $(\tilde{u}, \tilde{v}, \tilde{w})$ is the optimal pair for problem (P^ε) . This concludes the proof of Lemma 2.3. \square

Lemma 2.4 Let w_ε be optimal for problem (P^ε) and $(u_\varepsilon, v_\varepsilon)$ be the solution of (2.3) corresponding to w_ε . Then, on some subsequence ε_n ,

$$(u_{\varepsilon_n}, v_{\varepsilon_n}) \rightarrow (u^*, v^*) \quad \text{strongly in } (L^2(0, T; V) \cap C(0, T; H))^2, \quad (2.35)$$

$$w_{\varepsilon_n} \rightarrow w^* \quad \text{strongly in } L^2(0, T; U). \quad (2.36)$$

Proof Since w_ε is a solution to (P_ε) , we have

$$L_\varepsilon(w_\varepsilon) \leq \int_0^T [g_\varepsilon(t, u_\varepsilon^*) + h(w_\varepsilon^*)] dt + \frac{1}{2\delta(\varepsilon)} [d_S(F(u_\varepsilon^*)) + \delta(\varepsilon)]^2, \quad (2.37)$$

which together with (2.27) implies that

$$\begin{aligned} \frac{1}{2\delta(\varepsilon)} [d_S(F(u_\varepsilon^*)) + \delta(\varepsilon)]^2 &\leq \frac{1}{2\delta(\varepsilon)} [|F(u_\varepsilon^*) - F(u^*)|_Z + \delta(\varepsilon)]^2 \\ &\leq \frac{1}{2\delta(\varepsilon)} [C|u_\varepsilon^* - u^*|_{L^2(0, T; H)} + \delta(\varepsilon)]^2 \\ &\leq \frac{(1+C)^2}{2} \delta(\varepsilon) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \end{aligned} \quad (2.38)$$

which combined with (2.37) implies that

$$\limsup_{\varepsilon \rightarrow 0} L(w_\varepsilon) \leq \int_0^T [g(t, u^*) + h(w^*)] dt, \quad (2.39)$$

which implies that (2.39), that w_ε is bounded in $L^2(0, T; U)$. Without loss generality, we may assume that $w_\varepsilon \rightarrow \tilde{w}$ weakly in $L^2(0, T; U)$, which together with Lemma 2.2 implies that there exists a sequence of ε_n such that

$$(u_{\varepsilon_n}, v_{\varepsilon_n}) \rightarrow (\tilde{u}, \tilde{v}) \quad \text{strongly in } (L^2(0, T; V) \cap C(0, T; H))^2. \quad (2.40)$$

On the other hand, (2.28) and (2.39) imply that

$$\lim_{\varepsilon_n \rightarrow 0} d_S(F(u_{\varepsilon_n})) = 0, \quad (2.41)$$

and therefore

$$\lim_{\varepsilon_n \rightarrow 0} d_S(F(\tilde{u})) = 0. \quad (2.42)$$

Thus, we conclude from (2.28), (2.40) and (2.42) that

$$\liminf_{\varepsilon_n \rightarrow 0} L_{\varepsilon_n}(w_{\varepsilon_n}) \geq \int_0^T [g(t, \tilde{u}) + h(\tilde{w})] dt. \quad (2.43)$$

Finally, it follows from (2.39), (2.43) and Lemma 2.2 that

$$\begin{aligned} (u_{\varepsilon_n}, v_{\varepsilon_n}, w_{\varepsilon_n}) &\rightarrow (u^*, v^*, w^*) \\ &\text{strongly in } (L^2(0, T; V) \cap C(0, T; H))^2 \times L^2(0, T; U). \end{aligned} \quad (2.44)$$

This completes the proof. \square

3 The optimality condition for (P^ε) and (P)

In the following we derive the optimality condition for problem (P) by showing the relation between approximation problem (P^ε) and problem (P) . We start this section with the necessary conditions for w_ε to be optimal for (P^ε) .

Lemma 3.1 *Suppose that (H_1) , (H_2) , (H_3) and (H_4) hold. Let w_ε be optimal for problem (P^ε) and $(u_\varepsilon, v_\varepsilon)$ be the solution of (2.3) corresponding to w_ε . Then there exists a tetrad $(\mu_\varepsilon, p_\varepsilon, q_\varepsilon, \zeta_\varepsilon) \in R \times L^2(0, T; V) \cap C(0, T; H) \times L^2(0, T; V) \cap C(0, T; H) \times Z^*$ such that*

$$\begin{cases} -p_{\varepsilon,t} - \gamma \Delta p_\varepsilon + \beta'(u_\varepsilon)p_\varepsilon + (3u_\varepsilon^2 - 1)p_\varepsilon - \lambda''(u_\varepsilon)p_\varepsilon v_\varepsilon - \lambda'(u_\varepsilon)q_\varepsilon \\ \quad = -[\partial F(u^*)]^* \zeta_0 - \mu_0 \partial g(t, u^*), \\ -q_{\varepsilon,t} + \lambda'(u_\varepsilon)p_{\varepsilon,t} - k \Delta q_\varepsilon + \lambda''(u_\varepsilon)p_\varepsilon u_{\varepsilon,t} = 0, \\ p(T) = 0, \quad q(T) = 0 \end{cases} \quad (3.1)$$

$$B^* q_\varepsilon = \mu_\varepsilon [\nabla h(w_\varepsilon) + w_\varepsilon - w^*] \quad \text{a.e. } t \in [0, T] \quad (3.2)$$

and

$$\zeta_\varepsilon \in \partial d_S(F(u_\varepsilon)), \quad (3.3)$$

where ∂d_S is the sub-differential of d_S .

Proof Let $w \in L^2(0, T; U)$, $w_\varepsilon^\chi = w_\varepsilon + \chi w$ and $(u_\varepsilon^\chi, v_\varepsilon^\chi)$ be the solution of (2.3) corresponding to w_ε^χ . Then it is clear that

$$(u_\varepsilon^\chi, v_\varepsilon^\chi) \rightarrow (u_\varepsilon, v_\varepsilon) \quad \text{strongly in } C(0, T; H) \cap L^2(0, T; V) \text{ as } \chi \rightarrow 0. \quad (3.4)$$

Now, owing to the fact that w_ε is optimal for problem (P_ε) , we have $\frac{L_\varepsilon(w_\varepsilon^\chi) - L_\varepsilon(w_\varepsilon)}{\chi} \geq 0$ (for all $\chi > 0$), hence

$$0 \leq \mu_\varepsilon \int_0^T \langle \nabla g_\varepsilon(t, u_\varepsilon), y_\varepsilon \rangle + \langle \nabla h(w_\varepsilon) + w_\varepsilon - w^*, w \rangle_U dt + \langle (F'(u_\varepsilon))^* \zeta_\varepsilon, y_\varepsilon \rangle, \quad (3.5)$$

where $(y_\varepsilon, \bar{y}_\varepsilon)$ is the solution to

$$\begin{cases} y_{\varepsilon,t} - \gamma \Delta y_\varepsilon + \beta'(u_\varepsilon)y_\varepsilon + (3u_\varepsilon^2 - 1)y_\varepsilon - \lambda''(u_\varepsilon)y_\varepsilon v_\varepsilon - \lambda'(u_\varepsilon)\bar{y}_\varepsilon = 0, \\ \bar{y}_{\varepsilon,t} + \lambda'(u_\varepsilon)y_{\varepsilon,t} - k \Delta \bar{y}_\varepsilon + \lambda''(u_\varepsilon)y_\varepsilon u_{\varepsilon,t} = 0, \\ y_\varepsilon(0) = 0, \quad \bar{y}_\varepsilon(0) = 0. \end{cases} \quad (3.6)$$

Next, employing the same arguments as in the proof of [23], we conclude that

$$\lim_{\chi \rightarrow 0} \frac{1}{\chi} \int_0^T g_\varepsilon(t, u_\varepsilon^\chi) - g_\varepsilon(t, u_\varepsilon) dt = \int_0^T \langle \nabla g_\varepsilon(t, u_\varepsilon), y \rangle dt, \quad (3.7)$$

$$\begin{aligned} \lim_{\chi \rightarrow 0} \frac{1}{\chi} \int_0^T \left[(h(w_\varepsilon^\chi) - h(w_\varepsilon)) + \frac{1}{2} (|w_\varepsilon^\chi - w^*|_U^2 - |w_\varepsilon - w^*|_U^2) \right] dt \\ = \int_0^T \langle \nabla h(w_\varepsilon) + w_\varepsilon - w^*, w \rangle dt \end{aligned} \quad (3.8)$$

and

$$\lim_{\chi \rightarrow 0} ([d_S F(u_\varepsilon^\chi) + \varepsilon]^2 - [d_S F(u_\varepsilon) + \varepsilon]^2) = \frac{F(u_\varepsilon) + \varepsilon}{\varepsilon} \langle \zeta_\varepsilon, F'(u_\varepsilon) y \rangle_{Z^*, Z}, \quad (3.9)$$

where $\nabla g_\varepsilon(t, u_\varepsilon)$ denotes the gradient of g_ε to the second variable at u_ε and $\nabla h(w_\varepsilon)$ denotes the gradient of h at w_ε . Here, $\zeta_\varepsilon \in \partial d_S(F(u_\varepsilon))$ and ∂d_S is the sub-differential of d_S , which implies (3.3). Thanks to S being convex, closed and Z^* being strictly convex, we may also infer that

$$|\zeta_\varepsilon|_{Z^*} = \begin{cases} 1 & \text{if } F(u_\varepsilon) \notin S, \\ 0 & \text{if } F(u_\varepsilon) \in S. \end{cases} \quad (3.10)$$

Let

$$\mu_\varepsilon = \frac{\delta(\varepsilon)}{\delta(\varepsilon) + d_S(F(u_\varepsilon))} \quad (3.11)$$

and $(p_\varepsilon, q_\varepsilon)$ be the solution of (3.1). It follows from (3.1), (3.5) and (3.6) that

$$0 \leq \int_0^T -\langle B^* q_\varepsilon, w \rangle + \mu_\varepsilon \langle \nabla h(w_\varepsilon) + w_\varepsilon - w^*, w \rangle_U dt, \quad (3.12)$$

which implies (3.2). This completes the proof. \square

The proof of Theorem 1.1 By using the properties of α^ε and β^ε and Lemma 2.2, we have that, on a sequence of ε still denoted by ε ,

$$(u_\varepsilon, v_\varepsilon) \rightarrow (u, v) \quad \text{weakly in } (L^\infty(0, T; V) \cap L^2(0, T; H^2))^2, \quad (3.13)$$

$$(u_\varepsilon, v_\varepsilon) \rightarrow (u, v) \quad \text{strongly in } (C(0, T; H) \cap L^2(0, T; V))^2, \quad (3.14)$$

$$(u'_\varepsilon, v'_\varepsilon) \rightarrow (u', v') \quad \text{weakly in } (L^2(0, T; H))^2, \quad (3.15)$$

$$\beta'(u_\varepsilon) y_\varepsilon \rightarrow \eta \quad \text{weakly star in } (L^\infty(Q_T))^*. \quad (3.16)$$

It follows from (3.10) and (3.11) that

$$1 \leq \mu_\varepsilon + |\zeta_\varepsilon|_{Z^*} \leq 2 \quad \text{for any } \varepsilon > 0. \quad (3.17)$$

Therefore, there exist generalized subsequences of μ_ε and ζ_ε such that

$$\mu_\varepsilon \rightarrow \mu_0 \quad \text{as } \varepsilon \rightarrow 0 \quad (3.18)$$

and

$$\zeta_\varepsilon \rightarrow \zeta_0 \quad \text{weakly star in } Z^* \text{ as } \varepsilon \rightarrow 0. \quad (3.19)$$

Using Lemma 2.2, we may pass to the limit in (3.2) and derive (1.3)₁.

On the other hand, due to Lemma 2.2 and the same argument as in [23], we can conclude that

$$\nabla g_\varepsilon(t, u_\varepsilon) \rightarrow \rho(t) \quad \text{weakly in } L^2(0, T; H) \text{ as } \varepsilon \rightarrow 0, \quad (3.20)$$

where $\rho(t) \in \partial g(t, u^*)$ for all most $t \in (0, T)$. Thanks to (H_1) and Lemma 2.2, we also infer

$$[F'(u_\varepsilon)]^* \zeta_\varepsilon \rightarrow [F'(u^*)]^* \zeta_0 \quad \text{weakly in } L^2(0, T; H) \text{ as } \varepsilon \rightarrow 0. \quad (3.21)$$

Now we claim that

$$(3u_\varepsilon^2 - 1)p_\varepsilon \rightarrow (3(u^*)^2 - 1)p \quad \text{weakly star in } L^2(0, T; V^*) \text{ as } \varepsilon \rightarrow 0. \quad (3.22)$$

Indeed, let $w \in L^2(0, T; V)$ and $v = \max\{|p|_{L^2(0, T; V)} + 1, |u^*|_{L^2(0, T; V)} + 1, |w|_{L^2(0, T; V)}\}$, then we derive

$$\begin{aligned} & \int_0^T |(3u_\varepsilon^2 - 1)p_\varepsilon - (3(u^*)^2 - 1)p, w| dt \\ & \leq \int_0^T |\langle 3(u_\varepsilon^2 - (u^*)^2)p + [3(u_\varepsilon^2 - 1)(p_\varepsilon - p)], w \rangle| dt \\ & \leq \int_0^T |\langle 3(u_\varepsilon + u^*)(u_\varepsilon - u^*)p_\varepsilon + [3(u_\varepsilon^2 - 1)(p_\varepsilon - p)], w \rangle| dt \\ & \leq 2(v^6 + 1) \left[\max_{0 \leq t \leq T} |u_\varepsilon - u^*|_2^2 + \int_0^T |p_\varepsilon - p|_2^2 dt \right] \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned} \quad (3.23)$$

With the help of (3.13), (3.16), (3.18)-(3.23), we can pass to the limit in (3.1) to derive that $(p, q) \in (L^2(0, T; V) \cap C(0, T; H))^2$ and satisfies (1.2). On the other hand, observing that $\zeta_\varepsilon \in \partial d_S(F(u_\varepsilon))$, we derive

$$\langle \zeta_\varepsilon, w - F(u^*) \rangle_{Z^*, Z} \leq \langle \zeta_\varepsilon, F(u^*) - F(u_\varepsilon) \rangle_{Z^*, Z}, \quad (3.24)$$

which together with (3.19) and Lemma 2.2 implies (1.3)₂ (the second inequality of (1.3)).

Finally, we are in a position to prove that $(\mu_0, \zeta_0) \neq 0$. To this end, we suppose that $\mu_0 = 0$. It follows from (3.17), (3.19) and (3.24) that there exist $\varepsilon_1 > 0$ and $\delta > 0$ such that

$$\delta \leq |\zeta_\varepsilon|_{Z^*} \leq 2 \quad \text{for any } \varepsilon < \varepsilon_1 \quad (3.25)$$

and

$$\langle \zeta_\varepsilon, w - F(u^*) \rangle_{Z^*, Z} \leq \langle \zeta_\varepsilon, F(u^*) - F(u_\varepsilon) \rangle_{Z^*, Z} \rightarrow 0 \quad \text{uniformly in } w \in S. \quad (3.26)$$

Since $S \subset X$ is a closed convex subset with finite co-dimensionality, so is $S - F(u^*)$, which together with (3.25) and (3.26) implies that $(\mu_0, \zeta_0) \neq 0$ [24].

Assuming that $F'(u^*)$ is injective and $(\mu_0, p, q) = 0$, thanks to (1.2), we derive $(F'(u^*))^* \zeta_0 = 0$, which yields $\zeta_0 = 0$ and $(\mu_0, \zeta_0) = 0$. This is a contradiction with $(\mu_0, \zeta_0) \neq 0$. Thus, if $F'(u^*)$ is injective, then $(\mu_0, p) \neq 0$. We complete the proof. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

JZ carried out the optimal control problem for the phase-field transition system with state constraint and obstacle and drafted the manuscript. HL and JL participated in the design of the study and examined the results carefully. All authors read and approved the final manuscript.

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Received: 23 June 2013 Accepted: 25 September 2013 Published: 08 Nov 2013

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10.1186/1687-2770-2013-234

Cite this article as: Zheng et al.: State-constrained optimal control of phase-field equations with obstacle. *Boundary Value Problems* 2013, **2013**:234